Transparency Assumption in Hypersonic Radiative Gas dynamics

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A detailed analytical investigation is presented of the influence of radiant energy transport on the structure of the inviscid flow field in the stagnation region of a blunt body in hypersonic flight. It is shown that the classical condition of small characteristic optical depth is not sufficient to justify neglecting self-absorption in flows with strong radiation. The omission of self-absorption terms leads to a qualitatively incorrect description of inviscid flow temperature and density distributions in the strongly cooled region near the body streamline and results in a zero temperature at this streamline. The inclusion of self-absorption eliminates these anomalous behavior characteristics. A second-order approximate solution is given which includes at each point local self-absorption of radiation emitted by hotter gas in the neighborhood of the point. This solution exhibits a reasonable physical behavior throughout the flow field and predicts that gas temperature approaches a well-defined nonzero value at the body surface. This provides an enthalpy potential for convective surface heat transfer in flow regimes where viscous effects are confined to a thin boundary layer. Direct estimates are thus obtained of the influence of radiation energy transport on both radiative and convective heat transfer.

Nomenclature

a = exponent, $a = 4/n$
b = optical thickness parameter, Eq. (34)
C = specific heat, dimensionless
d = exponent, Eq. (24)
E = blackbody emissive power, $E = \sigma T^4$
$\epsilon_n(\tau)$ = exponential integral function $= \int_1^\infty \omega^{-n} \exp(-\omega\tau)\, d\omega$
h = specific enthalpy
K = const, Eq. (42)
k = const, Eq. (34)
l = parameter in exponential approximation, Eq. (33)
m = exponent, Eq. (34)
n = exponent, Eq. (20)
p = pressure
q_c = convective heat-transfer rate
q_{c0} = convective heat-transfer rate without radiation cooling
q_r = radiative heat flux
T = temperature
u = velocity in x direction
X = coordinate normal to body surface
x = stretched coordinate normal to surface
Y = tangential coordinate
α = exponent, Eq. (23)
β = function defined by Eq. (18)
Γ = radiation strength parameter, Eq. (5a)
Γ' = classical gamma function
δ = shock standoff distance with radiation absent
δ_r = shock standoff distance of radiating shock layer
$\varkappa$ = const, Eq. (34)
ξ = dummy variable
μ = radiation absorption coefficient
η = perturbation parameter, Eq. (30)
ρ = density
σ = Stefan-Boltzmann constant
τ = optical depth
ζ = transformed enthalpy variable, Eq. (26)

Superscript

$(\bar{\ })$ = quantity normalized with respect to conditions immediately behind shock.

Subscripts

∞ = freestream
s = condition immediately behind shock

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Introduction and Summary

THE effect of radiative energy transport on the structure of the flow field about a body in hypersonic flight increases strongly with increasing flight velocity. For flight conditions expected during many missions presently under serious study, e.g., atmospheric entry from a Mars-Earth transfer orbit, radiation effects are of extreme importance.

The role of radiation in hypersonic flow problems is usually discussed in terms of the dimensionless parameter Γ that is the ratio of the rate of radiation of energy by shock-heated gas to the rate of convection of energy into the shock layer. When $\Gamma \ll 1$, radiation has a negligible influence on the structure of the flow field and only contributes to the surface energy transfer. Goulard[1,2] has shown that the radiation field is significantly altered by radiative losses when $\Gamma \sim 0.01$ and has estimated the resulting effect on stagnation-point radiation energy transfer for the limiting case of an optically thin (transparent) shock layer. Much larger values of Γ are possible in applications such as the Mars mission mentioned previously. Numerical solutions by Wilson and Hoshizaki[3] exhibit profound changes in the temperature and density distributions and in the shock standoff distance of a transparent, inviscid shock layer when Γ becomes of order 1 or greater. These authors also point out a fundamental consequence of the transparency assumption that allows no self-absorption of radiation; the gas temperature everywhere along the body streamline is forced to approach zero when emission from the wall is neglected and to approach a value less than wall temperature when wall emission is included. Goulard's asymptotic solutions for small Γ also exhibit this same behavior.

The classical criterion for the validity of the transparency assumption in radiative gasdynamics is that the optical depth, based on a characteristic dimension of the gas region, be small. This criterion has been used by previous investigators as a basis for neglecting self-absorption of radiation in hypersonic flow problems. It is the purpose of the present analysis to demonstrate that this criterion is not sufficient when applied to flows with strong radiation cooling, such as the examples mentioned previously. In such flows, self-absorption may be highly significant. This is particularly true in the most strongly cooled regions of the fluid, where absorption of energy emitted by hotter regions may represent a dominant contribution to the local internal energy density. Furthermore, the inclusion of self-absorption eliminates anomalous behavior characteristics such as the zero temperature body streamline in inviscid flow.

This latter point has important implications with respect to the manner of computing convective energy transfer to the body surface. With the gas temperature near the wall forced to approach a value less than wall temperature, the temperature potential for convective energy transfer to the wall is removed. It is then meaningless to apply the classical approach of using the wall conditions from an inviscid flow field calculation as edge-of-boundary-layer conditions for computing the convective heat transfer. On purely physical grounds, however, there is nothing in the nature of radiative energy transport (when properly accounted for) which would indicate this classical approach to be invalid for flight regimes where the viscous boundary layer is thin compared to the shock standoff distance. The present problem is, therefore, a singular perturbation problem analogous to the ordinary fluid dynamic boundary layer; the optical thickness, however small, cannot be completely neglected, if one is to obtain a solution that is uniformly valid throughout the shock layer.

This paper describes a detailed analytical investigation of the influence of radiant energy transport on the structure of the inviscid flow field in the stagnation region of a blunt-nosed body in hypersonic flight. Approximate solutions are obtained for various levels of approximation in treating the radiative transport. The zeroth-order approximate solution allows local emission only and corresponds to the limit of a completely transparent gas. This solution is equivalent to Goulard's result.[2] It exhibits a physically unacceptable behavior in the region near the body streamline. A higher-order approximate solution is obtained which includes at each point local self-absorption of radiation emitted by hotter gas in the neighborhood of the point. This solution exhibits qualitatively correct physical behavior throughout the flow field and predicts that gas temperature approaches a well-defined nonzero value at the body surface.

It is concluded that the classical condition of small characteristic optical depth is not sufficient to justify neglecting self-absorption of radiation in flows with strong radiation cooling even though the self-absorption terms are at most of second-order in optical depth. In such cases, a detailed examination of the relative contribution of emission and self-absorption terms in the energy conservation equation is necessary.

Analysis

The present investigation is concerned with radiative effects on the structure of the inviscid region of the shock layer on a blunt body in hypersonic flight. In the stagnation region, the one-dimensional flow model introduced by Goulard[1,2] is applicable (Fig. 1). In this model it is assumed

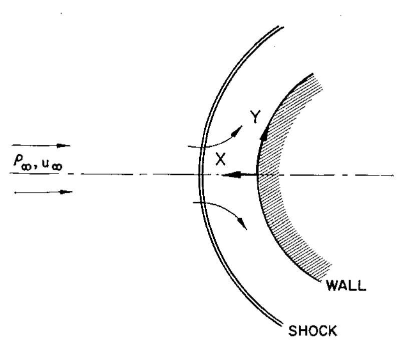

Fig. 1 Shock layer model.

that, as in stagnation flow without radiation, the temperature is independent of Y to within terms of order Y^2:

$$T(X, Y) = T(X) + O(Y^2)$$

The Mach number behind the shock is small; hence both dynamic pressure and kinetic energy are negligible compared to the local static pressure and static enthalpy, respectively:

$$\rho V^2/2 \ll p \qquad V^2/2 \ll h$$

The static pressure is therefore taken to be constant across the layer. The enthalpy and density then vary only as a result of radiative energy transport within the gas.

Neglecting molecular transport processes, the energy conservation equation takes the form

$$\rho u(dh/dX) = -\mu(dq_r/d\tau) \tag{1}$$

where the radiative flux is given by[4]

$$q_r = \int_0^\tau 2E\epsilon_2(\tau - \xi)d\xi - \int_\tau^{\tau_s} 2E\epsilon_2(\xi - \tau)d\xi + 2E_w\epsilon_3(\tau) \tag{2}$$

assuming a black wall at temperature T_w. The optical depth is measured from the wall:

$$\tau = \int_0^X \mu dX$$

Note that in the absence of radiation, $q_r = 0$, the enthalpy and density are uniform across the shock layer, and the shock standoff distance δ is known.[5] Following Goulard,[2] we nondimensionalize Eqs. (1) and (2), using as reference conditions the corresponding variables in the radiationless shock layer. There results

$$\bar{\rho}\bar{u}(d\bar{h}/d\bar{x}) = -[(\Gamma/4)\bar{\mu}(d\bar{q}_r/d\tau)] \tag{3}$$

where

$$\bar{q}_r = \int_0^\tau 2\bar{E}\epsilon_2(\tau - \xi)d\xi - \int_\tau^{\tau_s} 2\bar{E}\epsilon_2(\xi - \tau)d\xi + 2\bar{E}_w\epsilon_3(\tau) \tag{4}$$

$$\Gamma = 4\mu_s\delta E_s/\rho_\infty u_\infty h_s \tag{5a}$$

$$\bar{X} = X/\delta \tag{5b}$$

Goulard[2] has discovered a coordinate transformation that approximately accounts for density variations induced by the radiative transport†:

$$dx/dX = \rho^{1/2} \tag{6}$$

The transformed distance x is identified with the axial coordinate in the constant density flow, which would exist in the absence of radiation. The mass flux normal to the surface then takes the same form as in constant density stagnation flow:

$$\rho u = -x \tag{7}$$

The locations of wall and shock in the stretched coordinate system are, respectively,

$$x = 0 \quad \text{at} \quad X = 0 \tag{8a}$$

$$x = 1 \quad \text{at} \quad X = \delta_r/\delta \tag{8b}$$

where δ_r is the shock standoff distance in the radiative flow. From Eq. (6),

$$\frac{\delta_r}{\delta} = \int_0^1 \rho^{-1/2}\, dx \tag{9}$$

Inserting Eqs. (6) and (7) into (3) and performing the indicated differentiation on the flux equation [Eq. (4)], we obtain

$$x\frac{dh}{dx} = \frac{\Gamma}{2}\frac{\mu}{\rho^{1/2}}\left[2E - \int_0^{\tau_s} E\epsilon_1(|\tau - \xi|)d\xi - E_w\epsilon_2(\tau)\right] \tag{10}$$

† The bars are suppressed. All of the variables are understood as dimensionless throughout the remainder of the paper.

that is an integrodifferential equation for h, since, at constant pressure, both the density and the absorption coefficient are functions of a single thermodynamic variable, say enthalpy. The boundary condition is imposed at the shock

$$h(x = 1) = 1 \tag{11}$$

Written in terms of the stretched coordinate, the optical depth is

$$\tau = \mu_s\delta \int_0^x \frac{\mu}{\rho^{1/2}}\, dx \tag{12}$$

The right-hand side of Eq. (10) represents the effect of radiative energy transport on the internal energy of the gas. The first term in square brackets accounts for the energy loss by emission, the second term represents self-absorption of radiation emitted elsewhere in the gas, and the last term represents absorption of energy radiated by the wall. The presence of the self-absorption integral makes Eq. (10) intractable except by numerical methods. We will examine accordingly various levels of approximation in treating this term.

Zeroth-Order Approximation

The simplest course is to neglect self-absorption entirely. This may be accomplished by passing to the limit of a completely transparent gas, i.e., by taking $\tau_s \to 0$ in the right-hand side of Eq. (10). One then obtains

$$dh/dx = (\Gamma/2)(\mu/\rho^{1/2})[(2E - E_w)/x] \tag{13}$$

Note that, in this transparent limit, the emission term is preserved, as is the term representing absorption of external radiation, but the self-absorption term vanishes. That is, emission is a first-order effect in optical depth, whereas self-absorption is second-order.

For $T_w = 0$, Eq. (13) is identical to the equation used by Goulard[2] in studying radiation cooling effects for small Γ. The equation is separable and has been integrated by Goulard.[2] As remarked previously, the solution is reasonable everywhere except in the neighborhood of the wall, where it indicates that the temperature vanishes.

This result may be deduced directly from the differential equation itself by examining the topological structure of the integral curves near the origin of the x-E plane. Introducing first the dimensionless specific heat

$$C = (\partial h/\partial T)_{p=1} = C_pT_s/h_s$$

which may be an arbitrary function of the local thermodynamic state, Eq. (13) may be written as

$$dE/dx = 2\Gamma(\mu E^{3/4}/\rho^{1/2}C)[(2E - E_w)/x] \tag{14}$$

This equation has critical points at $(x, E) = (0, 0)$ and $(x, E) = (0, E_w/2)$, both of which are nodes. The boundary conditions are applied at the point $(x, E) = (1, 1)$. Equation (14) implies that at this point, as in the entire first quadrant, the slope of the solution curve is positive $dE/dx > 0$ as long as the wall is cooler than the shock. Referring to Fig. 2, the solution curve therefore begins at the point (1, 1) and proceeds downward and to the left, toward the origin. The solution curve cannot cross the solid line $E = E_w/2$ because on that line the solution must have zero slope from Eq. (14), whereas the slope must be nonzero, if the line is to be crossed. Similarly, the integral curve cannot, in general, touch the line $x = 0$, since that line is a locus of points at which the slope of the solution curve must be infinite. The solution curve is therefore forced toward the critical point $(x, E) = (0, E_w/2)$ where, from Eq. (14), the slope dE/dx is indeterminate.‡ Physically, this simply means that the gas reaches the wall, $x = 0$, with a temperature less than wall temperature:

$$\lim_{x\to 0} T = T_w/2^{1/4}$$

‡ A detailed asymptotic analysis of the topological structure of the field of integral curves near the node reveals that, as long as $T_s > T_w/2^{1/4}$, all of the solutions must approach the node with infinite slope $dE/dx = +\infty$.

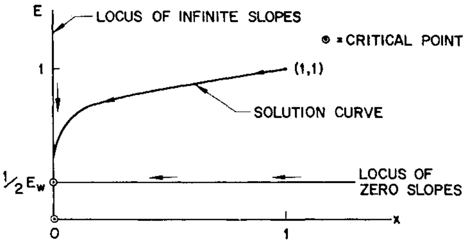

Fig. 2 Topology of solution curves for zeroth order, solution, Eq. (14).

The physical explanation for this behavior is that the gas loses energy continuously by emission to all space as it flows along the stagnation streamline. It only absorbs energy radiated from the half-space occupied by the wall. Since an infinite time is required for the gas to reach the wall, equilibrium is achieved, but at a temperature below wall temperature. Note that this anomalous behavior holds true regardless of the strength of the radiation, i.e., even for $\Gamma \ll 1$ but nonzero.

First-Order Approximation

The physically unacceptable behavior of the transparent-gas solution near the wall results from neglecting self-absorption. This effect must, therefore, be included, at least approximately. An estimate of the self-absorption term may be obtained by expanding the integrand $E(\xi)$ in a Taylor series about $\xi = \tau$

$$E(\xi) = E(\tau) + (\xi - \tau)(dE/d\xi)_{\xi=\tau} + \ldots \tag{15}$$

Truncating the series (15) and substituting the result into Eq. (10), various levels of approximation may be obtained, depending upon the number of terms retained.

Neglecting all of the terms of Eq. (15) except the first gives the "local temperature" approximation proposed by Goulard for the case where the optical depth is appreciable, $\tau_s \sim 0.3$. Equation (10) again becomes a differential equation:

$$\frac{dE}{dx} = 2\Gamma\, \frac{\mu E^{3/4}}{\rho^{1/2}C} \left\{\frac{E[\epsilon_2(\tau) + \epsilon_2(\tau_s - \tau)] - E_w\epsilon_2(\tau)}{x}\right\} \tag{16}$$

The topological method used earlier may be employed directly to investigate the behavior of the solution near the wall without actually integrating the equation.

As for the zeroth-order approximation, Eq. (16) has two critical points; one at the origin and one at the point:

$$(x, E) = \{0, E_w/[1 + \epsilon_2(\tau_s)]\}$$

The topological structure of the field of integral curves is identical to that discussed earlier for the zeroth approximation, except that the isocline of zero slopes is the curve

$$E/E_w = \epsilon_2(\tau)/[\epsilon_2(\tau) + \epsilon_2(\tau_s - \tau)]$$

Hence, the solution curve must touch asymptotically the critical point as the wall is approached, and the gas temperature at the wall is

$$\lim_{x\to 0} T = T_w/[1 + \epsilon_2(\tau_s)]^{1/4}$$

for all nonzero values of the radiation strength parameter Γ.

The preceding first-order solution is also physically unrealistic in the region near the wall. The reason is that the self-absorption term is estimated on the basis of the local temperature alone. That is, the gas at each point is allowed

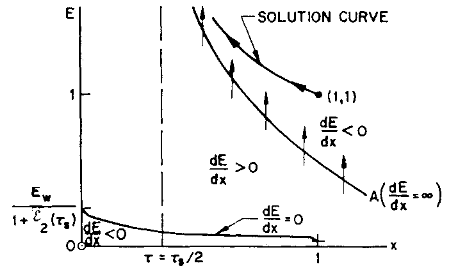

Fig. 3 Topology of solution curves for second order, solution, Eq. (17), when $\Gamma > -n\mu_s\delta/2\beta(\tau_s)$.

to absorb only radiation emitted at the local temperature. This is a poor approximation in relatively cool regions where the locally emitted intensity is small compared to the intensity of radiation emitted by hotter gas in neighboring regions. This suggests that a reasonable estimate of the self-absorption term must include some mechanism, which allows a variation in emission between neighboring regions of the gas. The local gradient term of the Taylor series (15) provides such a mechanism.

Second-Order Approximation

If the second term of the series (15) is retained in estimating the self-absorption term, the differential equation (10) assumes the form

$$\frac{dE}{dx} = 2\Gamma \frac{\mu E^{3/4}}{\rho^{1/2}C} \left\{ \frac{E[\epsilon_2(\tau) + \epsilon_2(\tau_s - \tau)] - E_w\epsilon_2(\tau)}{x + [2\Gamma E^{3/4}\beta(\tau)/\mu_s\delta C]} \right\} \quad (17)$$

where

$$\beta(\tau) = \tau\epsilon_2(\tau) + \epsilon_3(\tau) - (\tau_s - \tau)\epsilon_2(\tau_s - \tau) - \epsilon_3(\tau_s - \tau) \quad (18)$$

Equation (17) is a first-order equation, and we may again determine the asymptotic behavior of the solution near the wall by examining the topological structure of the integral curves near the line $x = 0$ of the x-E plane (Figs. 3 and 4). There is again a critical point at the origin, but the other critical point present in the zeroth and first approximations has been removed by inclusion of the second-order self-absorption term. The isocline of zero slopes of the integral curves is the same as that of the first-order approximation. The isocline of infinite slopes is no longer the E axis but rather the curve

$$E^{3/4}/C = -\mu_s\delta x/2\Gamma\beta(\tau) \quad (19)$$

In general, over restricted ranges of temperature, enthalpy and density may be approximated adequately by power-law functions

$$h = T^n \qquad \rho = T^{-n} \qquad \text{at } p = \text{const} \quad (20)$$

For a perfect gas, $n = 1$. At the higher temperatures of interest here, $T > 8000°K$, Eqs. (20) with $n = \frac{5}{3}$ fit Gilmore's equation of state data[6] reasonably well. The left-hand side of Eq. (19) is therefore an increasing function of E:

$$E^{1-n/4} = -n\mu_s\delta x/2\Gamma\beta(\tau) \quad (21)$$

It may be verified easily that β is a monotonically decreasing function of τ having a zero at $\tau = \tau_s/2$. Hence, $\beta > 0$ for $0 \leq \tau < \tau_s/2$, and $\beta < 0$ for $\tau_s/2 < \tau \leq 1$. Equation (21) represents a double-branched curve. One branch lies in the lower half plane for $x > 0$, passes through the origin, and is asymptotic to the vertical line $\tau = \tau_s/2$. The second branch lies in the upper half plane and is asymptotic to the positive x axis and to the vertical line $\tau = \tau_s/2$ (Fig. 3). Since $E = T^4$ is positive definite, only the upper branch is of physical interest; call it curve A. Examination of Eq. (17) reveals that all of the integral curves lying above curve A must have negative slope $dE/dx < 0$, whereas below curve A, the integral curves have positive slope $dE/dx > 0$. The boundary condition is applied at $(x, E) = (1, 1)$. The vertical position of curve A depends upon the numerical value of the radiation strength parameter Γ. If $\Gamma > -n\mu_s\delta/2\beta(\tau_s)$, curve A lies below the boundary point (1, 1). The integral curve originating at (1, 1) must then have a negative slope at that point. Such an integral curve can never cross curve A in the region $0 \leq x \leq 1$ because, arbitrarily near curve A, the slope of the integral curve becomes negative and large. Proceeding in toward the wall, $x = 0$, the solution curve must remain above curve A. There can exist no bounded solutions to Eq. (17) in this case.

In the case where Γ is sufficiently small,

$$\Gamma < -n\mu_s\delta/2\beta(\tau_s) \quad (22)$$

Curve A lies above the boundary value point, and the integral curve passing through this point represents a bounded, monotonic solution in the entire shock layer, $0 \leq x \leq 1$. In particular, we note that the line $x = 0$ is no longer an isocline of infinity; hence the solution curve may intersect this line at any point in the range

$$E_w\epsilon_2(\tau_s)/[1 + \epsilon_2(\tau_s)] \leq E \leq 1$$

Physically, this means that the gas temperature approaches a well-defined nonzero value at the wall, and this value depends upon both the optical depth τ_s and on the radiation strength parameter Γ.

The second-order approximation yields physically reasonable solutions for all of the values of Γ which satisfy inequality (22). Wilson and Hoshizaki[3] have shown that the optical thickness of the shock layer is generally small in practice, $\tau_s \leq \mu_s\delta \leq 0.1$. Inequality (22), therefore, holds even for $\Gamma \sim 10$ or higher which includes most situations of practical interest. The failure of the second-order solution for very large Γ is undoubtedly caused by the omission of higher-order terms in the Taylor series (15).

Of course, the second-order solution itself provides no information on its absolute accuracy. This presumably could be obtained only by comparison with an exact solution. However, we have demonstrated that the inclusion of self-absorption to second-order eliminates the physically unacceptable behavior of the lower-order solutions in the region near the wall. We therefore proceed with this level of approximation and seek a solution to Eq. (17). The dimensionless absorption coefficient is a function of temperature alone, since pressure is assumed uniform across the shock layer. We assume a power-law variation

$$\mu = T^\alpha \quad (23)$$

Using this along with Eq. (20), Eq. (17) may be written in terms of enthalpy alone:

$$\frac{dh}{dx} = \frac{\Gamma}{2} h^{a+d} \frac{\epsilon_2(\tau) + \epsilon_2(\tau_s - \tau)}{x + (2\Gamma/n\mu_s\delta)h^{a-1}\beta(\tau)} \quad (24)$$

where

$$a = 4/n \qquad d = \alpha/n + \tfrac{1}{2}$$

and the wall emission term has been neglected.

Equation (24) is still basically a type of integrodifferential equation because of the integral definition of optical depth [Eq. (12)]. Integrating Eq. (12) by parts, one obtains

$$\frac{\tau}{\mu_s\delta} = x\frac{\mu}{\rho^{1/2}} - \int_0^x x\, d\frac{\mu}{\rho^{1/2}}$$

The left-hand side of this equation is positive, and $\mu/\rho^{1/2}$ is a monotonically increasing function of T and therefore of x for the temperatures of interest. The second term on the right is, therefore, smaller than the first and will be neglected

$$\tau/\mu_s\delta \approx x[\mu(x)/\rho^{1/2}(x)] \approx xh^d(x) \quad (25)$$

With this approximation, Eq. (24) becomes strictly a first-order differential equation for h. Transforming to the new dependent variable

$$\zeta = 1/(a + d - 1)\Gamma h^{a+d-1} \tag{26}$$

gives Eq. (24) the somewhat simpler form

$$\frac{d\zeta}{d\tau} = -\frac{1}{2}\left[\frac{\epsilon_2(\tau) + \epsilon_2(\tau_s - \tau)}{\tau + 2\beta(\tau)/n(a + d - 1)\zeta}\right] \tag{27}$$

The transformed boundary condition (11) becomes

$$\zeta(\tau = \tau_s) = \zeta(x = 1) = \zeta_1 \tag{28}$$

where

$$\zeta_1 = 1/(a + d - 1)\Gamma \tag{29}$$

Asymptotic solution for $\Gamma \ll 1$

It is possible to obtain an explicit asymptotic solution to the second-order approximation [Eq. (27)] when the radiation strength parameter is small. In this case, the radiation will cause only small deviations from the uniform temperature and density distributions that would exist, if no radiative transport occurred. The relative deviation may be written as

$$\eta(\tau) = [\zeta(\tau) - \zeta_1]/\zeta_1 \tag{30}$$

Substituting Eq. (30) into Eq. (27) and linearizing the resulting differential equation for $\eta \ll 1$, one obtains

$$\frac{d\eta}{d\tau} = -\frac{1}{2\zeta_1}\left[\frac{\epsilon_2(\tau) + \epsilon_2(\tau_s - \tau)}{\tau + 2\beta(\tau)/n(a + d - 1)\zeta_1}\right] \tag{31}$$

With the boundary condition $\eta(\tau_s) = 0$, Eq. (31) integrates to

$$h^{-(a+d-1)} \sim 1 + \frac{a + d - 1}{2}\Gamma \int_\tau^{\tau_s}\left[\frac{\epsilon_2(\tau) + \epsilon_2(\tau_s - \tau)}{\tau + (2\Gamma/n)\beta(\tau)}\right]d\tau \tag{32}$$

The preceding solution is valid for arbitrary optical thickness, but the integral must be evaluated numerically. A considerable simplification is obtained by applying the Schuster-Schwarzschild approximation. This is equivalent to replacing the exponential integral functions by simple exponentials

$$\epsilon_2(\tau) \approx (l^2/3)\exp(-l\tau) \tag{33a}$$

$$\epsilon_3(\tau) \approx (l/3)\exp(-l\tau) \tag{33b}$$

The criterion for selecting the parameter l is somewhat arbitrary. We simply choose $l = (3)^{1/2}$ in the present application so that the radiative flux at the wall, $\tau = 0$, will be properly represented. It was noted earlier that $\tau_s < 0.1$ for most applications. Expanding the exponentials of Eq. (33) and retaining only terms of lowest order in optical depth gives the following differential equation:

$$d\zeta/dx = -\{\varkappa/\zeta^m + x[1 - (k/\zeta)]\}^{-1} \tag{34}$$

where

$$m = (a - 1)/(a + d - 1)$$

$$k = [2b/(a + d - 1)](\mu_s\delta/\tau_s)$$

$$b = (l^3\mu_s\delta/3n)(\tau_s/\mu_s\delta)^2$$

$$\varkappa = \Gamma b/[\Gamma(a + d - 1)]^m$$

Applying the transformation (30), linearizing the result, and integrating in the same manner as was done for Eq. (27) yields the following expression for the enthalpy distribution:

$$h \sim \{1 + (a + d - 1)\Gamma\log[(1 + \Gamma b)/(x + \Gamma b)]\}^{-1/(a+d-1)} \tag{35}$$

As indicated by our investigation of the topological structure of the field of integral curves, the solution is well behaved throughout the shock layer. In particular, the enthalpy approaches a well-defined and physically reasonable value at the wall, $x = 0$.

The enthalpy distribution (35) reduces precisely to Goulard's solution[2] in the transparent limit, $\tau_s \to 0$. For τ_s arbitrarily small but nonzero, Eq. (35) goes smoothly to the proper uniform shock layer solution in the limit as the radiation strength parameter $\Gamma \to 0$, whereas Goulard's result displays a discontinuous behavior.

The relation between the actual physical coordinate X and the stretched coordinate x may be obtained directly from Eq. (6):

$$X \sim x\left(1 - \frac{\Gamma}{2}\right) + \frac{\Gamma}{2}\log\left\{\left(\frac{x + \Gamma b}{1 + \Gamma b}\right)^{x+\Gamma b} \Big/ \left(\frac{\Gamma b}{1 + \Gamma b}\right)^{\Gamma b}\right\} \tag{36}$$

The change in shock standoff distance resulting from the density perturbation, induced by radiation cooling, is given by Eq. (9) or equivalently by Eq. (36), evaluated at $x = 1$, as

$$\delta_r/\delta \sim 1 - (\Gamma/2)\{1 - \Gamma b\log[1 + (1/\Gamma b)]\} \tag{37}$$

The ratio of the radiative heat transfer at the wall to that which would be received from a uniform temperature shock layer is given by Eq. (4) evaluated at $\tau = 0$. For small optical depth, $\tau_s \ll 1$, this expression reduces to

$$\frac{q_r}{2\mu_s\delta\sigma T_s^4} = \int_0^1 E\left(\frac{\mu}{\rho^{1/2}}\right)dx = \int_0^1 h^{a+d}\,dx \tag{38}$$

which, for the enthalpy distribution (35), integrates to

$$q_r/2\mu_s\delta\Gamma T_s^4 \sim 1 - (a + d)\Gamma\{1 - \Gamma b\log[1 + (1/\Gamma b)]\} \tag{39}$$

Note that the actual optical depth of the shock layer τ_s that appears in the constant b is unknown because of its integral definition, Eq. (12). Upon inserting Eqs. (20, 23, and 35) into Eq. (12), one finds that the ratio $\tau_s/\mu_s\delta$ differs from unity by a quantity, which is of order Γ

$$\tau_s/\mu_s\delta \sim 1 - O(\Gamma)$$

Substituting this into any of Eqs. (35–39) introduces additional terms, which are, at most, of order (Γ^2). Hence, use of the approximation $\tau_s/\mu_s\delta \approx 1$ in evaluating the constant b is consistent with all of the assumptions used to obtain the asymptotic solutions (35–39) to first order in Γ.

General solution for arbitary Γ

Under conditions where the radiation strength parameter is no longer small, it is not likely that a general analytic solution can be obtained to Eq. (17). For the case of small optical depth, however, the differential equation assumes the much simpler form given in Eq. (34). Regarding x as the dependent variable, the equation is linear and may be integrated with the aid of the integrating factor $\zeta^{-k}e^{\zeta}$ to

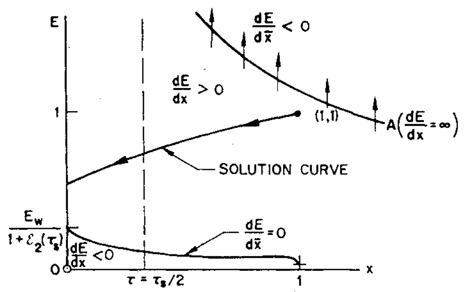

Fig. 4 Topology of solution curves for second order, solution, Eq. (17), when $\Gamma < -n\mu_s\delta/2\beta(\tau_s)$.

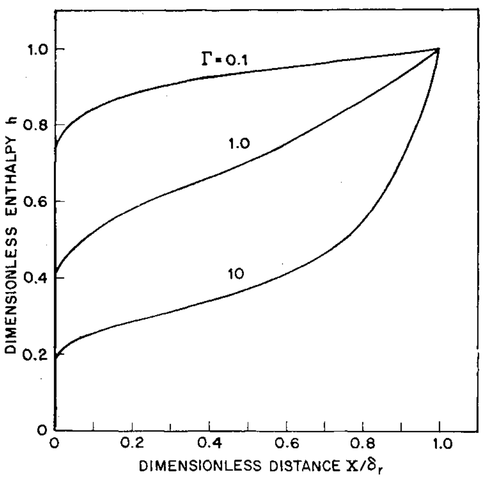

Fig. 5 Typical enthalpy profiles, $\mu_s\delta = 0.1$, $d = 1.6$.

obtain an implicit solution $x(\zeta)$. Under the boundary condition [Eq. (28)], the solution may be written as

$$x\zeta^{-k}e^{\zeta} - \zeta_1^{-k}e^{\zeta_1} = \varkappa \int_{\zeta}^{\zeta_1} \zeta^{-(k+m)}e^{\zeta}d\zeta \tag{40}$$

Performing successive operations of integration by parts on the right-hand side of Eq. (40) gives a uniformly convergent series that may be recognized as a confluent hypergeometric function. The solution (40) may then be written in the form

$$\zeta^{-k}e^{\zeta}[K\mathrm{x} + \varkappa\zeta^{1-m}{}_1F_1(1;\ 1 + K;\ -\zeta)] = \zeta_1^{-k}e^{\zeta_1}[K + \varkappa\zeta_1^{1-m}{}_1F_1(1;\ 1 + K;\ -\zeta_1)] \tag{41}$$

where, in the notation of Erdelyi,[7] the confluent hypergeometric function is

$${}_1F_1(1;\ 1 + K;\ -\zeta) = \sum_{i=0}^{\infty} \frac{\Gamma'(1 + K)}{\Gamma'(1 + K + i)} (-\zeta)^i \tag{42}$$

where Γ' is the complete gamma function of classical function theory, and

$$K = 1 - (k + m) = (d - 2b)/(a + d - 1)$$

Equation (41) is essentially an implicit solution for the enthalpy distribution in the shock layer and is valid for all of the values of the radiation strength parameter which satisfy inequality (22). As for the asymptotic solution obtained earlier, the enthalpy approaches a well-defined and physically reasonable value at the wall. This value is given by the solution to the transcendental algebraic equation obtained by setting $x = 0$ in Eq. (41).

Numerical Results

The final solution, Eq. (41), has been evaluated numerically for a wide range of cases of practical interest. Results are displayed in Figs. 5–9 for two separate power-law curve fits to Planck mean absorption coefficient data, corresponding to the lower and upper bound emissivity estimates of Wilson and Hoshizaki.[3] The corresponding exponents in the curve fit are $d = 1.6$ for the lower bound and $d = 3.75$ for the upper bound, respectively. These values give a reasonable fit to the emissivity data for shock layer pressures p_s of order 1 atm. The exponent in the approximate equation of state (20) was taken to be $n = \frac{5}{3}$, which fits Gilmore's data[6] rather well at the high enthalpy levels of major interest. The parameter $\tau_s/\mu_s\delta$ appearing in the constants k and b is not known a priori; precise numerical results were therefore obtained by iteration on its value.

Typical enthalpy profiles for weak, moderate, and strong radiation are shown in Fig. 5. A marked change in character of the profile occurs as the radiation strength parameter becomes large. In general, the profiles in the stretched coordinate system x were observed to be identical to those predicted by the exact solution with no self-absorption [Eq. (35) with $\tau_s = 0$] except in a small region near the wall, where the error in (35) is large. Note that, in the solutions of Fig. 5 which contain self-absorption, the enthalpy approaches a value at the wall that is a substantial fraction of the shock value.

The enthalpy at the wall for both the lower and upper bound emissivity is plotted in Figs. 6 and 7 as a function of the radiation strength for a range of values of the optical depth parameter, $\mu_s\delta$. It is remarked that the actual optical thickness of the shock layer τ_s is considerably smaller than the value $\mu_s\delta$ when Γ is large, because of the strong radiation cooling effect. The shock standoff distance is also reduced, as shown in Fig. 8. The radiation losses also have a very profound effect on the radiative heat transfer to the surface [Eq. (38)], plotted in Fig. 9. It was observed that the effect of self-absorption on the surface radiant heat rate is small except for very strong radiation, $\Gamma \sim O(10)$. Figure 9 therefore contains a single curve for each set of emissivity data. Note the very good agreement of the present simple analytical theory with the numerical solutions of Wilson and Hoshizaki[3] for a transparent gas.

Discussion

It may be concluded from the previous analysis and results that the classical condition of small characteristic optical depth is not sufficient to justify neglecting self-absorption of radiation in flows with strong radiation cooling even though the self-absorption terms are at most of second order in optical depth. In the shock layer problem, the omission of self-absorption terms leads to a qualitatively incorrect description of inviscid flow temperature and density distributions in the strongly cooled region near the body streamline. The gas temperature in this region is erroneously forced to approach a value below wall temperature at the body streamline. This erroneous result influences the radiant energy transfer to the wall and has important implications with respect to the coupling between radiation

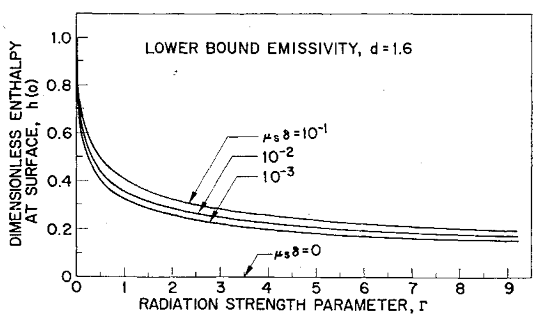

Fig. 6 Effect of radiation on gas enthalpy at body surface.

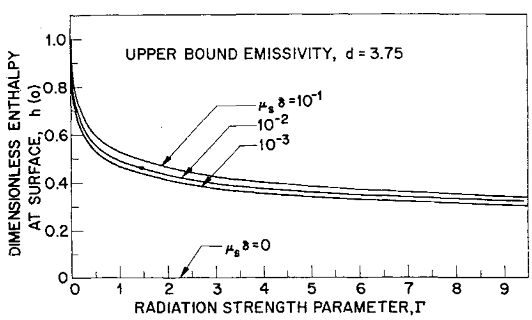

Fig. 7 Effect of radiation on gas enthalpy at body surface.

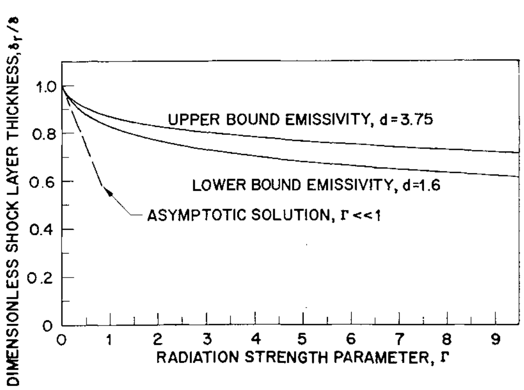

Fig. 8 Effect of radiation on shock standoff distance.

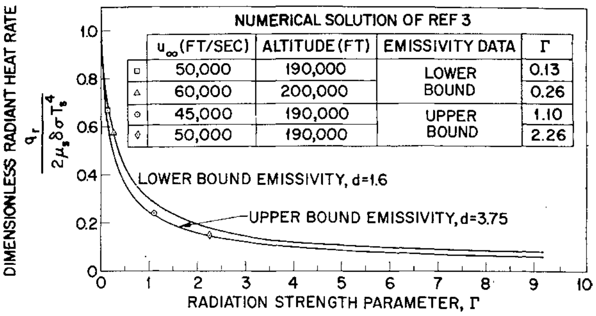

Fig. 9 Stagnation-point radiative heat transfer.

and convective surface heat transfer in flow regimes where viscous effects are confined to a thin boundary layer. The convective heat transfer cannot be estimated by the classical method of using the wall conditions from the inviscid flow as edge-of-boundary-layer conditions.

Goulard[1] circumvented the difficulty of the zero temperature body streamline in his solution and computed surface convective energy transfer in a somewhat arbitrary way. He used as the edge-of-boundary-layer condition the inviscid flow field enthalpy at a distance from the wall equal to the boundary-layer thickness. This may be a reasonable approach for the small values of Γ for which his solution is intended but becomes invalid for large Γ or for bodies of large nose radius.

The reason lies in the fact that the thickness of the region in which the transparent-gas solution gives large errors in the enthalpy distribution is magnified with increasing Γ. The boundary-layer thickness is fixed. Hence, as Γ is increased, the edge-of-boundary-layer enthalpy diverges rapidly from the correct value for a self-absorbing gas. This effect is particularly acute, because the enthalpy distribution in the transparent-gas solution of Ref. 2, or equivalently Eq. (35) with $\tau_s = 0$, has infinite slope at $x = 0$. Small errors in judging the position of the boundary-layer edge therefore may result in extremely large errors in predicted enthalpy potential. Similar errors will be obtained for bodies having large nose radii of curvature. The shock layer thickness is proportional to R, whereas the boundary-layer thickness increases only as $(R)^{1/2}$. Hence, for fixed Γ, the edge-of-boundary-layer temperature must decrease strongly with increasing nose radius. This behavior is clearly unreasonable.

In contrast, the present solution is well behaved in the entire shock layer, and the preceding difficulties do not arise. As a first approximation, one can compute the reduction in cold-wall convective energy transfer directly from the nondimensional enthalpy at the wall

$$q_c/q_{c_0} \approx h(0) = (\zeta_1/\zeta)^{1/(a+d-1)} \qquad (43)$$

The reduction in convective heat transfer because of radiation cooling may therefore be estimated directly from Figs. 6 and 7, or from Eq. (37) when $\Gamma \ll 1$. An indication of the usefulness of this estimate may be obtained by comparison with Howe's numerical solution for a viscous, radiating shock layer.[8] For a body of 5-ft nose radius with $u_\infty = 50{,}000$ fps at 190,000-ft altitude, Howe found

$$q_c/q_{c_0} = 0.46$$

Using his emissivity data, $d \approx 1.6$, $\Gamma = 0.386$, and $\mu_s\delta = 0.0294$. From Fig. 6,

$$q_c/q_{c_0} \approx 0.5$$

which is within 8% of Howe's result.

The preceding estimates of the reduction in convective heat transfer caused by radiation cooling assume that the detailed structure of the boundary layer is not significantly affected when the inviscid flow field is altered by radiative transport, that is, the usual boundary-layer solutions are assumed valid, and the only effect of the radiation is to reduce the enthalpy potential across the boundary layer. This is not quite correct even though radiative energy transport may be negligible compared to molecular conduction within the boundary-layer itself. The reason is that the radiative transport induces in the inviscid flow a substantial enthalpy gradient in the direction normal to the wall. The classical boundary-layer solutions are not applicable because they require a zero enthalpy gradient at the boundary-layer edge. There is, therefore, an interaction between the boundary layer itself and the external inviscid flow through this radiation-induced enthalpy gradient.

Vincenti[9] has pointed out the analogy between this situation and the interaction between the velocity boundary-layer and shock-generated vorticity in the external flow. The solution of such problems requires going beyond classical boundary-layer theory. Thus, aside from simple estimates such as those obtained here, a great deal more work will be necessary before the effect of radiation on convective heat transfer can be predicted with precision.

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